Financial Innovation, Communication and the Theory of the Firm

Marc Oliver Bettzüge and Thorsten Hens*†

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Abstract

When markets are incomplete, the competitive equilibria considered so far are not constrained Pareto–efficient, production efficiency breaks down and shareholders no longer agree on the objective function of the firm.

We first show by way of an example that these inefficiencies can result from the double role of firms in incomplete markets: providing high market value and providing good hedging opportunities (spanning role).

To disentangle these two conflicting roles of the firm's decision, we then suggest to let the firm choose a relevant financial policy by issuing securities being collaterized by the production plan. In order to guarantee that the firm does not choose to innovate trivial assets, it is then shown to be crucial that the firm‘s shareholders agree on the same set of state prices. Therefore we introduce some communication network into the model which allows the shareholders to exchange their views on the firm's best policies. In our main result we demonstrate that competitive equilibria with communication of shareholders and a relevant financial policy of the firm are Pareto–efficient, provided there are at least as many firms as there are shareholders.

Keywords: theory of the firm, incomplete markets, communication, financial innovation.

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†Institute for Banking and Finance University of Zurich, Plattenstrasse 32, CH-8032, Zurich, Switzerland, thorsten.hens@bf.uzh.ch

1 Introduction

The integration of productive activity into general equilibrium models has posed a major challenge ever since the work of Walras. The classic Arrow-Debreu-model has provided a solution to this task, provided that markets are complete (see e.g. Debreu (1959)). Subsequently, the Arrow-Debreu-model has been extended to the case of incomplete markets. Surveys of this topic can be found e.g. in Geanakoplos (1990), Magill and Shafer (1991), and Hens (1998). Moreover, Magill and Quinzii (1996) is an excellent text book thereon. In spite of the success of the incomplete markets model, however, a satisfactory treatment of the theory of the firm in this model has not yet been given. In fact, the approaches to this topic suggested so far imply that most of the standard efficiency properties derived for the general equilibrium model with complete markets are breaking down.

When markets are complete any two commodities (available in possibly different time periods and possibly contingent on certain events) can be compared by their present value. In this case the obvious objective function of the firm is to maximize the present value of its production. As a consequence, competitive equilibria are Pareto–efficient and in particular the production of any one firm cannot be raised without lowering the production of some other firm, i.e. 'production efficiency' holds. Moreover, the firm's production decisions are independent from their shareholders' preferences (i.e. the Fisher–Separation–Principle holds).

In the incomplete markets model, consumers will no longer agree on the present value of those payoffs that cannot be hedged on the existing financial markets. Hence, without further assumptions, the shareholders of a firm will not agree upon the choice of a production plan. Having realized this, one can either take some organizational form of the firm as given and then look into the (in)-efficiency properties of this organization or one could take a normative approach and try to find the organizational form that is most desirable according to some efficiency considerations. An interesting paper based on the first approach is DeMarzo (1993) who shows that, for generic economies, a majority rule equilibrium for a firm implies that production is optimal for the largest, or dominant, shareholder. We will follow the second – the normative – approach here. Based on some efficiency considerations Drèze (1974) suggested to evaluate a firm's production plan according to the average present value vector of its shareholders, where the weights in averaging are the shares the consumers hold. If consumers are not allowed to trade shares of the firms, and if there is a single consumption good in each state, this criterion leads to constrained Pareto-efficiency, i.e. competitive equilibria cannot be improved upon by a planner who has to use the exogenously given incomplete system of financial markets. Hence although shareholders do not agree about the optimal production plan the Drèze–criterion as the objective function for the firm yields the best efficiency result one could hope for in the presence of incomplete markets. Transferring this criterion to the more general case including trade on stock markets (Drèze (1974) and Grossman and Hart (1979)) the resulting competitive equilibria are no longer constrained Pareto–efficient and shareholders no longer agree that market value maximization should be the unique aim of a firm.

In this paper, we show that these conceptual problems could arise from the double role of the firm's production plan when markets are incomplete: providing high market value and providing good hedging opportunities (spanning role of a firm's production decisions)[1]. To disentangle these two conflicting roles we suggest to let the firm choose a relevant financial policy by which it issues securities being collateralized by its production plan. The choice of these securities is based on the spanning needs of its shareholders[2]. To be precise, the new securities issued by the firm are chosen according to the average vector of its shareholders' complete markets excess demand, where, as usual, the weights in averaging are the shares of the consumers. The production plan is chosen exactly in the same spirit, i.e. firms maximize the average present value of their production. In both decisions, the market value and the spanning decision, following Grossman and Hart (1979), averages are taken according to the initial shares of the consumers.

Note that in contrast to the standard financial policy of the firm (which consists of trading on a given set of financial markets), the financial policy we suggest is not *irrelevant* in the sense of Modigliani and Miller[3]. Furthermore note that in contrast to some recent literature on financial innovation the security design decision in our model is rather simple. It is directly based on the shareholders' spanning needs and it does not involve any anticipation of the consequences which alternative security designs will have for the shareholders' utility. For approaches of financial innovation relying on anticipation of induced changes in the competitive equilibrium see Duffie and Rahi (1996), Allen and Gale (1994) and Bisin (1998), for example. Furthermore in our model the security design is not based on any additional market imperfections like transaction costs or oligopolistic competition. Such imperfections interfere with the desired efficiency properties of competitive equilibria. The paper closest in spirit to our notion of financial innovation is Citanna and Vilanacci (1996) where an exchange economy is modeled in which every consumer can issue one asset without incurring any costs.

When agents have different state prices, it can occur in this set-up that firms choose to innovate trivial assets, i.e. not to innovate at all. In order to mitigate this effect we model a communication network by which they exchange their views on state prices, which we call beliefs. As in the choice of the financial policy of the firm we try to keep things simple and model communication by some fixed mechanistic process. According to this communication process, every agent's belief (on the profitability of the firms production plans) is obtained as an average of all those agents' beliefs with whom he communicates. For example, one could suppose that such communication takes place in the assembly of a firm's shareholders. Introducing non-market interactions like communication into a general equilibrium model, in which decisions are generally supposed to be taken independently from each other, might be regarded

[1]As is well known, there are many other reasons for Pareto-inefficiency in incomplete markets. E.g. with multiple commodities in each state trading assets can change the relative prices in the second period so as to create inefficiencies – see Geanakoplos and Polemarchakis (1986).

[2]Our idea to introduce relevant financial policy is in line with Shiller‘s (2012) manifest suggesting various ways how finance could serve better the society - one of which is issuing securities that people need to insure their most important risks. Having firms satisfying the insurance needs of their shareholders is a first step in this direction.

[3]For a the Modigliani-Miller Theorem in the type of models considered here see DeMarzo (1988)

as a surprising step. It becomes more evident, however, when taking a broader perspective on the literature. In game theory, for example, concepts of communication are used to solve problems of coordinating strategies in normal form games (Matsui (1991)). In general equilibrium theory coordination of net trades has been introduced in economies with externalities (Vind (1983)). In competitive financial markets De Marzo, Vayanos and Zwiebel (2003) have introduced communication of beliefs. In our model communication leads to a coordination of beliefs which in turn leads to a coordination of asset trades.

Our main result demonstrates that competitive equilibria with communication among shareholders and a relevant financial policy of firms are Pareto–efficient, provided there are at least as many firms as there are shareholders and provided some regularity conditions are met which rule out degenerated cases both in the communication network as well as in the ownership structure of firms. We show that this result is tight in the sense that without communication or with less firms than shareholders a planner who can anticipate the equilibrium consequences of financial innovations can Pareto–improve the competitive equilibrium by choosing a better financial policy for the firms.

This paper is organized as follows. Section 2 introduces the general model and notation. Section 3 then presents different forms of market structures for this model. We begin with the well known market structure of contingent contracts and then advance to the case of incomplete markets with stock markets. In section 4 we point out by way of an example why stock market economies are in general constrained Pareto-inefficient. In section 5 we propose our new equilibrium concept with relevant financial policies and communication, and in section 6 we prove that this concept restores Pareto-efficiency. We also show in this section why the introduction of a sufficiently rich communication network is essential to the results obtained. Section 7 then concludes the paper.

2 The Economy

There are two time periods, $t = 0, 1$, in each of which a single commodity is available. This commodity should be thought of as expenditure for multiple commodities that are not explicitly modeled here. There are $s = 1, \dots, S$ states of the world at $t = 1$.

Let $GE = [\mathbb{R}^{S+1}, (\mathcal{Y}^k)_{k=1,\dots,K}, (\mathcal{X}^i, U^i, \omega^i, \delta^i)_{i=1,\dots,I}]$ be a general equilibrium model with

$\mathbb{R}^{S+1}$ as commodity space,
$\mathcal{Y}^k \subset \mathbb{R}^{S+1}$ being firm k's production set and
$\mathcal{X}^i := \mathbb{R}^{S+1}_+$ being consumer i's consumption set,
$U^i : \mathcal{X}^i \to \mathbb{R}$ as consumer i's utility function.

Consumer i‘s endowments are given as
$\omega^i \in \mathcal{X}^i$ of commodities and
$\delta^i \in [0, 1]^K$ of shares of firms, where
$\sum\limits_i \delta^i_k = 1$, all $k = 1, \dots, K$.

In order to propose our new equilibrium concept we wish to avoid unneccesary technical problems. Therefore, we will assume that GE satisfies strong monotonicity, convexity and differentiability assumptions as for example in Magill and Shafer (1991):

Assumption 1 *(Agent characteristics) For every agent* $i \in \mathcal{I} = \{\infty, \ldots, \mathcal{I}\}$ *the following assumptions on utility functions and endowments are satisfied:*

(1) $U^i : \mathbb{R}_+^{S+1} \to \mathbb{R}$ *is continuous on* $\mathbb{R}_+^{S+1}$
and infinitely often differentiable on $\mathbb{R}_{++}^{S+1}$.
(2) If $\mathcal{U}^i(\xi) := \{x \in \mathbb{R}_+^{S+1} \mid U^i(x) \geq U^i(\xi)\}$ *then* $\mathcal{U}^i(\xi) \subset \mathbb{R}_{++}^{S+1}$, $\forall \xi \in \mathbb{R}_{++}^{S+1}$.
(3) For each $x \in \mathbb{R}_{++}^{S+1}$, *and for all* $h \neq 0$ *such that* $\nabla U^i(x)h = 0$ *it follows that* $\nabla U^i(x) \in \mathbb{R}_{++}^{S+1}$ *and* $h^T D^2 U^i(x) h < 0$.
(4) $\omega^i \in \mathbb{R}_{++}^{S+1}$.

An important characterization of GE-economies concerns the availability of markets for the trading of the $S+1$ commodities. As a major point of reference we therefore recall the well-known case of complete contingent contracts, i.e. the Arrow-Debreu-model. In this model it is assumed that there exists a market for every commodity $l = 1, \ldots, S+1$, and that on each of these markets a price π_l, $l = 1, \ldots, S+1$, is determined. In this situation, markets are said to be *complete.* In a *competitive equilibrium* for such an economy, every agent and every firm takes prices as given, consumers maximize utility, firms maximize market value and all markets clear:[4]

Definition

An Arrow-Debreu competitive equilibrium is an allocation $(\overset{*}{x}, \overset{*}{y}) \in \mathbb{R}^{(S+1)(I+K)}$ *and a price system* $\overset{*}{\pi} \in \mathbb{R}^{S+1}$ *such that*

1. $\overset{*}{y}{}^k \in \arg\max_{y^k \in \mathcal{Y}^k} \overset{*}{\pi}\, y^k$, *for every* $k = 1, \ldots, K$

2. $\overset{*}{x}{}^i \in \arg\max_{x^i \in \mathcal{X}^i} U^i(x^i)$, *for every* $i = 1, \ldots, I$

$$s.t.\ \overset{*}{\pi} \cdot x^i \leq \overset{*}{\pi} \cdot \omega^i + \sum_{k=1}^{K} \delta_k^i\, \overset{*}{\pi}\, \overset{*}{y}{}^k$$

3. $\sum_{i=1}^{I} \overset{*}{x}{}^i = \sum_{i=1}^{I} \omega^i + \sum_{k=1}^{K} \overset{*}{y}{}^k$.

In the incomplete markets model, in contrast, agents trade on sequential spot markets which are linked by an incomplete system of financial markets.

In order to allow agents to transfer wealth between the uncertain states $s = 1, \ldots, S$, there are financial assets with payoffs in period $t = 1$ which can be traded in the first period spot market. It is assumed that both exogenous assets and shares are

[4]In order to simplify our notation we use the usual economists' convention that quantities are denoted as column vectors and prices are denoted as row vectors.

available for trade. Hence, in the first period spot market, there are $j = 1, \dots, J$ assets whose payoffs $A^j \in \mathbb{R}^S$ are denominated in terms of the single commodity. Let $q_j,\ j = 1, \dots, J$ denote the price per unit of asset j being paid in $t = 0$. In addition to the existence of (exogenous) asset markets, agents can also trade shares in the firms on competitive stock markets. These stock markets also open in the first period of the two-periods model. Let r^k denote the sum of the k-th firm's stock market price and its first period investment $y_0^k,\ k = 1, \dots, K$. Then the matrix determining the possible income transfers can be written as $\begin{bmatrix} -q & -r \\ A & Y_1 \end{bmatrix}$, where we have defined $Y := (y^1, \dots, y^K)$, and $Y_1 := (y_1^1, \dots, y_1^K)$[5]. Note that consumer i's portfolio, θ^i, consists of shares of the exogenous assets and consumer i's net trade in the shares of the firms.

There is a well-known notion of a competitive equilibrium in this situation.

Definition Stock Market Competitive Equilibrium (FM)

A set of vectors $(\overset{*}{\theta}, \overset{*}{x}, \overset{*}{y}, \overset{*}{q}, \overset{*}{r}, \overset{*}{\pi}^k)$ *with* $\overset{*}{\pi}^k \begin{bmatrix} -\overset{*}{q} & -\overset{*}{r} \\ A & \overset{*}{Y}_1 \end{bmatrix} = 0$, *for every* $k = 1, \dots, K$ *is a financial markets competitive equilibrium for a stock market economy if*

1. $\overset{*}{y}^k \in \arg\max\limits_{y^k \in \mathcal{Y}^k} \overset{*}{\pi}^k y^k$ *for every* $k = 1, \dots, K$

2. $(\overset{*}{\theta}^i, \overset{*}{x}^i) \in \arg\max\limits_{x^i \in \mathcal{X}^i,\, \theta^i \in \mathbb{R}^{J+K}} U^i(x^i)$ *for every* $i = 1, \dots, I$

   $$s.t.\ (x^i - \omega^i - \sum_k \delta_k^i\, \overset{*}{y}^k) = \begin{bmatrix} -\overset{*}{q} & -\overset{*}{r} \\ A & \overset{*}{Y}_1 \end{bmatrix} \theta^i$$

3. $\sum\limits_i \overset{*}{x}^i = \sum\limits_i \omega^i + \sum\limits_k \overset{*}{y}^k$

4. $\sum\limits_i \overset{*}{\theta}^i = 0.$

In any stock market competitive equilibrium, asset and stock market prices must be arbitrage free, that is to say there should not exist any portfolio $\theta \in \mathbb{R}^{J+K}$ which delivers positive payoffs without requiring any investment, i.e. [6]

$$\nexists\, \theta \in \mathbb{R}^{J+K} \ : \ \begin{bmatrix} -\overset{*}{q} & -\overset{*}{r} \\ A & \overset{*}{Y}_1 \end{bmatrix} \theta > 0.$$

By a well known result in Linear Algebra the no arbitrage condition is equivalent

[5] For any vector $x \in \mathbb{R}^{S+1}$ we let $x_1 := (x_1, \dots, x_S) \in \mathbb{R}^S$

[6] We use the standard vector inequalities $x > y$ meaning $x_S \geq y_S$ all s and $x \neq y$.

to the existence of some strictly positive state price vector $\overset{*}{\pi} \in \mathrm{I\!R}^{S+1}_{++}$ such that

$$\overset{*}{\pi} \begin{bmatrix} -\overset{*}{q} & -\overset{*}{r} \\ A & \overset{*}{Y}_1 \end{bmatrix} = 0$$

Given any such belief $\overset{*}{\pi}{}^i$ one can rewrite the consumer's decision problem in the following way:

$$\begin{aligned} 2'. \qquad & \overset{*}{x}{}^i \in \arg\max_{x^i \in \mathcal{X}^i} U^i(x^i) \\ & \overset{*}{\pi}{}^i \cdot \ (x^i - \omega^i - \sum_k \delta^i_k \overset{*}{y}{}^k) = 0 \\ & (x^i_1 - \omega^i_1 - \sum_k \delta^i_k \overset{*}{y}{}^k_1) \in < A, \overset{*}{Y}_1 > . \end{aligned}$$

Moreover, in 2'. we can normalize state price vectors $\overset{*}{\pi}{}^i$ such that $\overset{*}{\pi}{}^i_0 = 1$. The competitive equilibrium defined above in its so called 'financial market version' (FM) can then equivalently be defined in the following 'no-arbitrage version' (NA).

Definition Stock Market Competitive Equilibrium (NA) *A set of vectors* $(\overset{*}{x}, \overset{*}{y}, \overset{*}{\pi}) \in \mathrm{I\!R}^{2(I+K)(S+1)}$ *with* $\overset{*}{\pi}{}^d_0 = 1$ *and* $(\overset{*}{\pi}{}^d_1 - \overset{*}{\pi}{}^{d'}_1)\left[A, \overset{*}{Y}_1\right] = 0$ *for all* $d, d' \in \{1, \ldots, I\} \cup \{1, \ldots, K\}$ *is a no–arbitrage competitive equilibrium for a stock market economy if*

1. $\overset{*}{y}{}^k \in \arg\max_{y^k \in \mathcal{Y}^k} \overset{*}{\pi}{}^k y^k$ *for every* $k = 1, \ldots, K$

2. $\overset{*}{x}{}^i \in \arg\max_{x^i \in \mathcal{X}^i} U^i(x^i)$ *for every* $i = 1, \ldots, I$

   s.t. $\overset{*}{\pi}{}^i \cdot \ (x^i - \omega^i - \sum_k \delta^i_k \overset{*}{y}{}^k) = 0$
   and $(x^i_1 - \omega^i_1 - \sum_k \delta^i_k \overset{*}{y}{}^k_1) \in < A, \overset{*}{Y}_1 >$

3. $\sum_i \overset{*}{x}{}^i = \sum_i \omega^i + \sum_k \overset{*}{y}{}^k.$

Without loss of generality, we will further on restrict attention to the (NA) definitions of equilibria.

3 Constrained Pareto-inefficiency of stock market equilibria

It is well known that competitive equilibria are *Pareto–efficient* when markets are complete, i.e. there does not exist an allocation $(\hat{x}, \hat{y})$ that is attainable ($\hat{x}^i \in \mathcal{X}^i, \hat{y}^k \in \mathcal{Y}^k$, $\sum_i x^i \leq \sum_i \omega^i + \sum_k y^k$) and satisfies $U^i(\hat{x}^i) \geq U^i(\overset{*}{x}{}^i)$ for all $i = 1, \ldots, I$ with one inequality being strict.

With incomplete markets the attainability notion has to be adapted to the asset span. In the notion of constrained Pareto–efficiency one compares the equilibrium allocations with alternative allocations that are attainable given the incomplete set of markets.

Definition Constrained Attainability (Stock Market Economy) *An allocation $(x, y) \in \mathrm{I\!R}^{(S+1)\times I} \times \prod_{k=1}^{K} \mathcal{Y}^k$ is constrained attainable, if $\sum_{i=1}^{I} x^i = \sum_{i=1}^{I} \omega^i + \sum_{k=1}^{K} y^k$ and $(x_1^i - \omega_1^i - \sum_k (\delta_k^i + \theta_k^i) y_1^k) \in < A, Y_1 >$ for every $i = 1, \ldots, I$.*

Constrained Pareto-efficiency is then straightforwardly defined as follows:

Definition Constrained Pareto–efficiency

An allocation (x, y) is constrained Pareto–efficient if there does not exist an alternative constrained attainable allocation $(\hat{x}, \hat{y})$ that satisfies $U^i(\hat{x}^i) \geq U^i(x^i),\ i = 1, \ldots, I$ with at least one inequality being strict.

If consumers were not allowed to trade the shares of the firms as financial assets, then it could easily be shown that the corresponding equilibria are constrained Pareto–efficient if the so-called "Grossman-Hart-criterion"

$$\pi^k = \sum_{i=1}^{I} \delta_k^i \nabla U^i(x^i).$$

is used for selecting the optimal production plan (see e.g. Bettzüge and Hens (2000)). According to this criterion, suggested by Grossman and Hart (1979), the firm should use the average of the consumers' (normalized) present value vectors, where the weights for averaging are the shares of the consumers.

In the stock market economy, however, this does not necessarily need to be true. To demonstrate why this is the case, we start with noting that the shares of firms are relevant as financial assets only if $< Y_1 >$ is not included in $< A >$, i.e. if these additional financial markets allow the agents to better finance their net trade on commodity markets. In particular, note that when asset markets are complete, i.e. when $< A > = \mathrm{I\!R}^S$, then there is no reason to trade on stock markets! When markets are incomplete, however, the choice of the production plan can have two effects on the consumers' budget set. As in the complete markets case the firm's market value $\overset{*}{\pi} \cdot \overset{*}{y}^k$ is part of the consumer's disposable income but in contrast to the complete markets case the choice of the production plan affects the spanning opportunities $< A, \overset{*}{Y}_1 >$.

This double role of the production plans implies that competitive equilibria of stock market economies no longer need to be constrained Pareto-efficient. Although the production plans in such equilibria still are profit maximal (part 1 of the definition) they might not be chosen such as to offer the optimal span of traded assets. In fact, since shares are traded assets, a benevolent planner now can freely determine up to K dimensions of the subspace of traded assets. As the following example demonstrates, his choice for the traded subset will generally not coincide with the one chosen by the notion of competitive equilibrium. This, of course, implies that we cannot expect competitive equilibria of a stock market economy to be constrained Pareto-efficient.

Example 1[7]

There are two states, $S = 2$, two consumers, $I = 2$, and one firm, $K = 1$. Each consumer is endowed with one unit of the commodity in the first period and consumer 1 (2) has one unit of the commodity in state 1 (2), i.e. $\omega^1 = (1, 1, 0)$, $\omega^2 = (1, 0, 1)$. Both consumers hold equal shares of the firm, i.e. $\delta^1 = \delta^2 = \frac{1}{2}$. Consumers do not value first period consumption and they evaluate second period consumption according to some expected utility function with the same objective probabilities $U^i(x_0^i, x_1^i, x_2^i) = u^i(x_1^i) + u^i(x_2^i)$ for $i = 1, 2$. On investing both units of the commodity available in the first period the firm can produce second-period output given by

$$\mathcal{Y}_1(\varepsilon) = \{(y_1, y_2) \mid \ y_1^2 + y_2^2 \leq 2\varepsilon^2\},$$

where we have fixed $y_0 = -2$. Figure 2 displays the second-period production possibility frontier as well as the corresponding Edgeworth-Box. When asset markets are complete, i.e. when $< A > = \mathbb{R}^2$, then the Pareto–efficient competitive equilibrium allocation is

$$\overset{*}{x}^1 = \overset{*}{x}^2 = \frac{1}{2}\begin{pmatrix} 0 \\ 1+\varepsilon \\ 1+\varepsilon \end{pmatrix}, \ \overset{*}{y} = \begin{pmatrix} -2 \\ \varepsilon \\ \varepsilon \end{pmatrix}.$$

Now suppose however, that markets are seriously incomplete because $< A > = \{0\}$. The firm's production plan, which is supposed to maximize its market value according to some strictly positive state prices, is strictly positive in the second period, i.e. $\overset{*}{y}_1(\varepsilon) \gg 0$. As an effect, the second-period components of the incomplete markets budget set collapse to the points $\omega_1^i + \frac{1}{2} < \overset{*}{y}_1 >$ for $i = 1, 2$. Hence when markets are

[7]Note that the agents' characteristics given in example 1 do not exactly satisfy the strong differentiability assumptions made in the presentation of the economy (utility of the agents does not vary with first period consumption, utilities are not defined on the boundary of the consumption sets, and the resources are not strictly positive). However, slightly perturbing the vector of resources and the utility functions would restore Assumption 1 without changing the results of Example 1.

incomplete there is no opportunity to trade and the equilibrium allocation is

$$\overset{*}{x}^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ \overset{*}{y}_1(\varepsilon) \\ \overset{*}{y}_2(\varepsilon) \end{pmatrix}, \quad \overset{*}{x}^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ \overset{*}{y}_1(\varepsilon) \\ \overset{*}{y}_2(\varepsilon) \end{pmatrix}.$$

Note that in both states $s = 1, 2$ *the optimal output* $\overset{*}{y}_s(\varepsilon)$ *is bounded above by* $\sqrt{2}\,\varepsilon$. *Hence for* $\varepsilon > 0$ *sufficiently small, the complete markets allocation Pareto–dominates the incomplete markets allocation. Moreover, a planner who is running the firm could implement the production plan* $\hat{y}(\varepsilon) = \begin{pmatrix} -2 \\ \varepsilon \\ -\varepsilon \end{pmatrix}$ *which is not market value maximizing but which offers perfect spanning opportunities. The resulting consumption allocation would be*

$$\hat{x}^1 = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ \varepsilon \\ -\varepsilon \end{pmatrix} = \hat{x}^2$$

which for $\varepsilon > 0$ *small enough Pareto–dominates the incomplete markets allocation. That is to say the competitive allocation is not constrained Pareto–optimal when markets are incomplete.*

$\square$

Note that the reasoning of Example 1 was done for any objective function of the firm that is exclusively based on the market value criterion. Similar examples have been given for a specific objective function (called the Drèze-criterion) which states that the firm uses the average present value vector of its *new* shareholders as the present value vector for profit maximization (Drèze (1974), Dierker, Dierker and Grodal (1999)). We therefore claim that the failure of constrained Pareto-efficiency to hold in stock market economies can be caused by the fact that the market value criterion is insufficient to take into account potential choices for the subspace of traded assets. In order to restore constrained Pareto-efficiency one has the following options:

1. Find a criterion for the selection of production plans which solves the inherent trade-off between spanning and market value maximization, and adjust part 1 of the definition of a competitive equilibrium accordingly.

2. Disentangle the production decision from the question which spanning opportunities are available in the economy.

While option 1 remains unsolved, we propose a solution to option 2 in the remainder of this paper, where we explicitly let the firms make two decisions: a financial decision which is relevant for spanning, and a production decision which maximizes the firms´ market value.

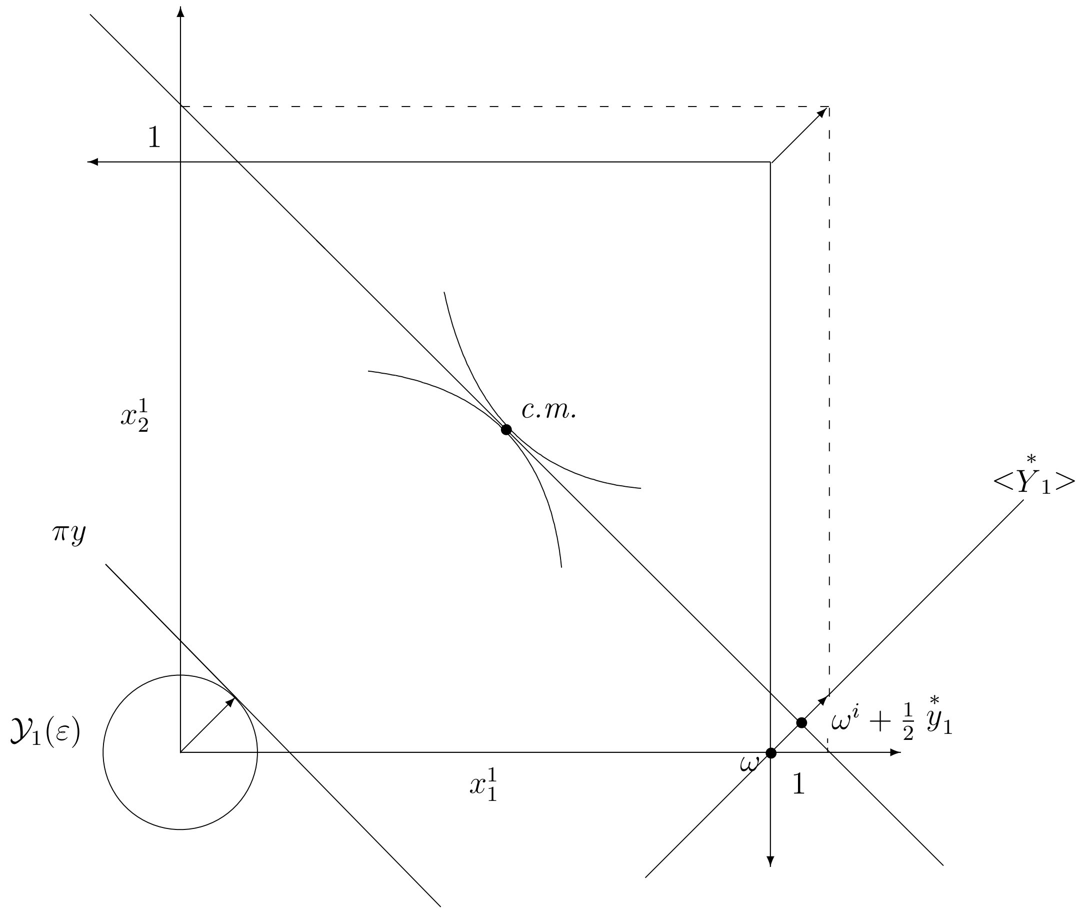

Figure 1: Edgeworth-Box illustrating the constrained Pareto-inefficiency of stock market equilibria in Example 1

4 A new objective function for the firm

4.1 Relevant financial policy

The objective functions for the firm which have so far been suggested in the literature are all based exclusively on the market value criterion. However, as our Example 1 demonstrates, maximizing market value can be in apparent contrast to the interests of the shareholders when the spanning role of the firm's decision becomes dominant. To serve these two aspects of a firm's decision we suggest to disentangle the market value aspect from the spanning aspect by allowing the firm to engage in a relevant financial policy. We think of the firm as having two departments, the production department and the finance department. Both departments are controlled by the assembly of its shareholders. The production department chooses a production plan which maximizes the firm's market value. In doing so it uses the Grossman–Hart–criterion which was doing fine in partnership economies where market value maximization was the only concern of the shareholders. The finance department can use the production plan y^k as a collateral for issuing a new security a^k which is supposed to satisfy the shareholders' spanning needs. In fact, we assume that instead of issuing the direct claim on its production plan y^k as a share, the firm issues *two* securities, a^k and $(y_1^k - a^k)$, both in net-supply of 1.[8] Note that this policy of issuing new securities is a financial policy which cannot be irrelevant in the sense of Modigliani and Miller. New financial securities will typically have a non-trivial effect on the marketed subspace, an effect which cannot be undone by the consumers using the existing assets. The firm's production plan y^k will typically be non-negative in the second period. Hence the firm is an institution which credibly can promise to deliver the period one payoffs of its financial security. This is the reason why firms play an important role as financial innovators. Of course these reasons are exogenous to the standard incomplete markets model without bankruptcy.[9]

The shareholders need to span their complete markets excess demand calculated at prices π^i which they take as given

$$
\begin{aligned}
z^i(\pi^i) := \arg\max_{z^i} U^i(\omega^i + \sum_k \delta^i_k y^k + z^i)& \\
s.t. \quad \pi^i \cdot z^i \leq 0& \\
(\omega^i + \sum_k \delta^i_k a^k + \sum_k \delta^i_k (y^k - a^k) + z^i) \in \mathcal{X}^i.&
\end{aligned}
$$

Of course different shareholders will have different spanning needs and again the firm averages those needs according to the shares of the consumers. This gives a simple rule for the financial innovation decision which reflects the consumers' power in the assembly of the firms' shareholders:[10]

$$a^k = \sum_{i=1}^{I} \delta^i_k z^i_1(\pi^i) \quad k = 1, \ldots, K.$$

Note that this criterion weighs the spanning interests of the old shareholders which is consistent with the Grossman-Hart criterion for the production decision. From now on we will assume that all financial assets $j = 1, \ldots, J$ are issued by firms, i.e. the market subspace $< A >$ consists of the linear space spanned by the columns $a^k, \; k = 1, \ldots, K$. Adding fixed securities different from a^k would not change our results but would unnecessarily complicate our exposition .

Based on these suggestions we can now define a competitive equilibirum in a stock market economy with relevant financial policy. Observe that in this definition we

[8]In this set-up, we imagine S, the set of possible states of the world, to be very large, and especially to be much larger than the number N of securities which could potentially be innovated by a single firm instead of its shares. For simplicity of the exposition but without loss of generality for the results, we then assume $N = 2$.

[9]For an incomplete markets model with bankruptcy see Dubey, Geanakoplos, Shubik (1997) and Zame (1993)

[10]Citanna and Villanacci (1996) consider an exchange economy where the consumers act as financial innovators, each of them generating the asset $a^i(\pi^i) = z^i(\pi^i)$ (using our terminology). Thus, the decision rule suggested by Citanna and Villanacci (1996) can be interpreted as the special case of the decision rule suggested here, when production sets are given by $\mathcal{Y}^k \subset \mathbb{R}^{S+1}_-$ for $k = 1, \ldots, K$, and each firm is owned by exactly one agent.

restate the firm's decision problem by choosing the alternative formulation where firm k issues asset y^k in net-supply of 1, and the asset a^k in zero net supply. This restatement has been done for consistency with previous definitions only, and does not affect our results.

Definition Stock Market Competitive Equilibrium with Relevant Financial Policy (NA)

A set of vectors (matrices, resp.) $(\overset{*}{x}, \overset{*}{y}, \overset{*}{A}, \overset{*}{\pi}{}^i)$ *with* $\overset{*}{\pi}{}^i \in R^{S+1}_{++}$, $\overset{*}{\pi}{}^i_0 = 1$ *and* $(\overset{*}{\pi}_1{}^i - \overset{*}{\pi}{}^d_1)[\overset{*}{A}, \overset{*}{Y}_1] = 0$ *for every* $i, d \in \{1, \ldots, I\}$ *is a no–arbitrage competitive equilibrium with relevant financial policy if*

1. $\overset{*}{y}{}^k \in \arg\max_{y^k \in \mathcal{Y}^k} (\sum_i \delta^i_k \, \overset{*}{\pi}{}^i) y^k, \quad \overset{*}{a}{}^k = \sum_i \delta^i_k \, \overset{*}{z}{}^i$ *for every* $k = 1, \ldots, K$ *where*

$$\begin{aligned} \overset{*}{z}{}^i \quad &= \quad \arg\max_{z^i} U^i(\omega^i + \sum_k \delta^i_k \, \overset{*}{y}{}^k + z^i), \quad i = 1, \ldots, I \\ s.t. \quad & \quad \overset{*}{\pi}{}^i \cdot \, z^i \le 0 \text{ and } (\omega^i + \sum_k \delta^i_k \, \overset{*}{y}{}^k + z^i) \in \mathcal{X}^i \end{aligned}$$

2. $\overset{*}{x}{}^i \in \arg\max_{x^i \in \mathcal{X}^i} U^i(x^i)$ *for every* $i = 1, \ldots, I$,
   s.t. $\overset{*}{\pi}{}^i \cdot \, (x^i - \omega^i - \sum_k \delta^i_k \, \overset{*}{y}{}^k) = 0$,
   and $(x^i_1 - \omega^i_1 - \sum_k \delta^i_k \, \overset{*}{y}{}^k_1) \in \langle \overset{*}{A}, \overset{*}{Y}_1 \rangle$

3. $\sum_i \overset{*}{x}{}^i = \sum_i \omega^i + \sum_k \overset{*}{y}{}^k$.

Some remarks will be useful in order to explain this equilibrium concept. Firstly, note that the equilibrium is *competitive* in the traditional sense, i.e. no agent makes any strategic conjectures about the way in which her actions will influence the equilibrium outcome. Consumer i simply takes the prices $\overset{*}{\pi}{}^i$, the production plans $\overset{*}{y}$, and the financial policies $\overset{*}{A}$ as given. Producers take the prices $\overset{*}{\pi}{}^i$ and the complete market demands $\overset{*}{z}{}^i$ of their shareholders as given. Secondly, observe that the consumers' state prices are treated as exogenous; hence, when markets are incomplete, the competitive equilibria defined above are indeterminate (cf. Duffie and Shafer (1988)).

The role of the state prices $\overset{*}{\pi}{}^i$ is analyzed in more depth in the following subsection. First, however, we will give an example which demonstrates that the competitive equilibria defined above are not necessarily constrained Pareto–efficient unless further restrictions are imposed. As in Example 1, constrained Pareto–inefficiency arises from the fact that the spanning opportunities might be inefficient. However, in contrast to Example 1, this inefficiency does not result from an inefficient trade-off between the production decisions. Rather, inefficiency here results from inefficient financial innovation chosen by the firms' financial departments.[11]

Example 2: *Consider an economy with I 'unproductive' firms, i.e. $\mathcal{Y}^k \subset \mathbb{R}^{S+1}_-$ for every $k = 1, \ldots, I$. Let $(\overset{*}{\pi}, z^i(\overset{*}{\pi}))$ be a competitive equilibrium for the corresponding Arrow–Debreu exchange economy, i.e. let*

$$\begin{aligned} \sum_i z^i(\overset{*}{\pi}) &= 0, \text{ where for all } i = 1, \ldots, I \\ z^i(\overset{*}{\pi}) &= \arg\max_{z^i} U^i(\omega^i + z^i), \\ s.t. \quad \overset{*}{\pi} \cdot z^i &\leq 0 \text{ and } (\omega^i + z^i) \in \mathcal{X}^i. \end{aligned}$$

Suppose that $U^i(\omega^i + z^i(\overset{*}{\pi}^i)) > U^i(\omega^i)$ for all $i = 1, \ldots, I$ and that

$$\delta^i_k = \begin{cases} 1 & k = i \\ 0 & k \neq i \end{cases}, \quad i, k = 1, \ldots, I.$$

This economy has at least the following two stock market competitive equilibria with relevant financial policy:

The first equilibrium results from the observation that $z^i(\nabla U^i(\omega^i)) = 0, \quad i = 1, \ldots, I$:

$$\begin{array}{ll} \overset{*}{x}^i = \omega^i, & i = 1, \ldots, I \\ \overset{*}{y}^k = 0, & k = 1, \ldots, K \\ \overset{*}{a}^k = 0, & k = 1, \ldots, K \\ \overset{*}{\pi}^i = \nabla U^i(\omega^i), & i = 1, \ldots, I \end{array}$$

The second equilibrium, however, is given by the following set of vectors:

$$\begin{array}{ll} \overset{*}{x}^i = \omega^i + \overset{*}{z}^i, & i = 1, \ldots, I \\ \overset{*}{y}^k = 0, & k = 1, \ldots, I \\ \overset{*}{a}^k = \overset{*}{z}^k, & k = 1, \ldots, I \\ \overset{*}{\pi}^i = \overset{*}{\pi}, & i = 1, \ldots, I \end{array}$$

Hence the second equilibrium Pareto–dominates the first.

□

The fact that firms choose an inefficient set of innovated assets is based on the lack of unanimity of the consumers' state prices $\overset{*}{\pi}^i$. Indeed, examples of this kind will persist in stock market competitive equilibria with relevant financial policy as long as agents' beliefs are heterogeneous. The following subsection will therefore introduce a framework to guarantee homogeneous beliefs in the economy.

[11] The following example illustrates an important result derived by Citanna and Villanacci (1996). They show that equilibria with various degrees of "market incompleteness" can coexist in an exchange economy where consumers also act as financial innovators: no innovation at all, innovation of an incomplete set of securities, and innovation of a complete set of securities.

4.2 Communication of beliefs

The role of beliefs

Introducing communication into interactive decision problems seems to be in conflict with the general equilibrium assumption that agents only interact on markets via prices. However, in general equilibrium theory a similar step has already been done by Vind (1983) who introduced coordination to solve the problem of externalities. Game theorists should be credited for having introduced the notion of *'preplay communication'* or *'cheap talk'* to solve some puzzles of non-cooperative game theory like the coordination on Pareto–efficient Nash–equilibria (c.f. Matsui (1991)). Similarly, in the general equilibrium model of this paper, communication can help to select Pareto–efficient equilibria. In fact, the fundamental problem of incomplete markets is that the prices $(\overset{*}{q}, \overset{*}{r})$ quoted by the auctioneer do not provide sufficient information how to evaluate an arbitrary payoff stream $y_1 \in \mathbb{R}^S$. While the evaluation of those components of y_1 lying in the marketed subspace is unambiguous, the evaluation of the complementary components cannot be inferred from the asset and stock market prices. At any equilibrium $(\overset{*}{x}, \overset{*}{y}, \overset{*}{A}, \overset{*}{\pi})$, however, every production plan *is* unambiguously priced because obviously then $\overset{*}{y}_1$ lies in the marketed subspace $<\overset{*}{A}, \overset{*}{Y}_1>$. In order to evaluate alternative production plans it would therefore be best to *know* the resulting equilibrium. This knowledge, of course, is hard to get. However agents might still form some *beliefs* about equilibrium prices. We therefore suggest to interpret $\overset{*}{\pi}^i$ as being consumer i's *belief* on the equilibrium state-prices.

Note that beliefs are based on the actual asset-stock-prices quoted by the auctioneer and that agents hold point expectations. Hence, given the belief $\overset{*}{\pi}^i$, agent i wants firm k to maximize the market value $\overset{*}{\pi}^i \cdot y^k$. Other agents may hold different beliefs and we assume that the decision of the assembly of shareholders will reflect their influence on the firm's market value objective, i.e. $\sum_i \delta^i_k \, \overset{*}{\pi}^i \cdot \, y^k$ is a natural objective function in this respect.

Consistency of beliefs

It is natural to assume that consumers meet other consumers (for example in the assemblies of the firms' shareholders), and that in these meetings they exchange their different views on the expected equilibrium state-prices. Hence in forming their beliefs consumers will be influenced by the beliefs of the other consumers. To ensure consistency between the individual beliefs, we propose the following rather simple consistency requirement:

$$\pi^i = \sum_{j=1}^{I} h^i_j \pi^j, \quad \text{for every } \; i = 1, \ldots, I,$$

where $h^i_j \geq 0$ denotes the weight agent i gives to agent j's belief. As a matter of normalization, $\sum_{j=1}^{I} h^i_j = 1$. Hence agents form their beliefs by taking convex combinations of all the other agents' beliefs. A special case arises if agents put strictly put positive weights only on those agents (including themselves) with whom they share a firm. In this case, the weights h are given by $h^i_j > 0$ if and only if there exists some $k \in K$ such that $\delta^i_k \delta^j_k > 0$.

We can now propose the following new equilibrium concept:

Definition Competitive Stock Market Equilibrium with Relevant Financial Policy and Communication (NA)

A set of vectors (matrices, resp.) $(\overset{*}{x}, \overset{*}{Y}, \overset{*}{A}, \overset{*}{\pi}{}^i)$ *with* $\overset{*}{\pi}{}^i \in \mathbb{R}^{S+1}_{++}$, $\overset{*}{\pi}{}^i_0 = 1$, $\overset{*}{\pi}{}^i = \sum_j h^i_j \overset{*}{\pi}{}^j$, *and* $(\overset{*}{\pi}{}^d - \overset{*}{\pi}{}^i)[\overset{*}{A}, \overset{*}{Y}_1] = 0$ *for every* $i, d = 1, \ldots, I$ *is a no–arbitrage competitive stock market equilibirum with relevant financial policy and communication if*

1. $\overset{*}{y}{}^k \in \arg\max_{y^k \in \mathcal{Y}^k} (\sum_i \delta^i_k \overset{*}{\pi}{}^i) y^k$, *and* $\overset{*}{a}{}^k = \sum_{i=1}^{I} \delta^i_k \overset{*}{z}{}^i$ *for every* $k = 1, \ldots, K$, *where*

$$\begin{aligned} \overset{*}{z}{}^i &= \arg\max_{z^i \in \mathbb{R}^{S+I}} U^i(\omega^i + \sum_k \delta^i_k \overset{*}{y}{}^k + z^i) \\ s.t. \quad & \overset{*}{\pi}{}^i \cdot z^i \leq 0 \text{ and } (\omega^i + \sum_k \delta^i_k \overset{*}{y}{}^k + z^i) \in \mathcal{X}^i \end{aligned}$$

2. $\overset{*}{x}{}^i \in \arg\max_{x^i \in \mathcal{X}^i} U^i(x^i)$*for every* $i = 1; \ldots, I$,
   s.t. $\overset{*}{\pi}{}^i \cdot (x^i - \omega^i - \sum_k \delta^i_k \overset{*}{y}{}^k) = 0$, *and*
   $(x^i_1 - \omega^i_1 - \sum_k \delta^i_k \overset{*}{y}{}^k_1) \in \langle \overset{*}{A}, \overset{*}{Y}_1 \rangle$

3. $\sum_i \overset{*}{x}{}^i = \sum_i \omega^i + \sum_k \overset{*}{y}{}^k$

In the 'communication network' we can think of every consumer as a node in a graph summarized by the following $I \times I$ matrix $H = \begin{bmatrix} h^1_1 & \ldots & h^I_1 \\ \vdots & \vdots & \vdots \\ h^1_I & \ldots & h^I_I \end{bmatrix}$. How useful for a competitive equilibrium the introduction of communication is will depend on the structure of this graph. For example, if there is no communication ($H = Id$), then beliefs are still exogenous in the new equilibrium notion .

We say consumer i is *'directed connected'* to consumer j if there is a chain of consumers $k_0 = i, k_1, k_2, ..., k_m = j : h^{k_n}_{k_{n+1}} > 0$. This defines a *'directed communication graph'*. Analogously one can define an undirected communication graph by saying i is *connected* to j if for some chain of consumers $k_0, \ldots, k_n \in I$ in any pair of neighbours k_j, k_{j+1} either $h^{k_n}_{k_{n+1}} > 0$ or $h^{k_{n+1}}_{k_n} > 0$.

Generally speaking, agents' beliefs will become more homogeneous the more the communication graphs are connected. And if two subsets of consumers are not connected then across those subjects beliefs can remain different.

The next proposition proves that beliefs will become homogeneous when the following notion of *'belief connectedness'* is satisfied:

Definition

The economy is belief connected *if there is some agent d to which every other agent is connected in the directed communications graph,i.e.:*

$$\exists\, d \in \mathcal{I} \ : \ \forall j \in \mathcal{I} \setminus \{d\} \ \exists\ j_0 = j, j_1, j_2, ..., j_m = d : h^{j_n}_{j_{n+1}} > 0\ \forall n = 0, 1, ..., m-1.$$

Hence a sufficient condition for homogenous beliefs is that there is some expert or "guru" whose beliefs are valued directly or indirectly by every other agent.

Proposition 1

If the economy is belief connected then the communication of beliefs leads to homogeneous beliefs.

Proof:

Consider any state $s = 0, \dots, S$. We will show that $\pi^i_s = \pi^j_s$ for all $(i, j) \in \mathcal{I} \times \mathcal{I}$. Define the vectors $\pi_s := (\pi^1_s, \dots, \pi^I_s) \in \mathbb{R}^{S+1}_{++}$. Then $\pi_s = \pi_s H$ from the consistency requirement with respect to state s. The fact that $\sum_j h^i_j = 1$ for every $i \in \mathcal{I}$ then implies that $\overset{*}{\pi}_s = \lambda(1, \dots, 1)$ for some $\lambda \in \mathbb{R}_{++}$ is a strictly positive solution to this system of equations.

We need to show that $\overset{*}{\pi}_s$ is the *unique* solution. To this end we show that $(Id - H)$ has rank $I - 1$, where Id denotes the identity matrix of dimension I. Note that belief connectedness implies:

$$\exists d \in \mathcal{I} \ : \ \forall J \subset \mathcal{I} \setminus \{d\}\ \exists j \in J \text{ with } h^j_i > 0 \text{ some } i \in \mathcal{I} \setminus J.$$

This claim follows from the following argument:

Let $J \subset \mathcal{I} \setminus \{d\}$. Pick any $j \in J$. Then by the assumption of belief connectedness there exists a sequence $j_0 = j, j_1, j_2, ..., j_m = d,$ such that $h^{j_n}_{j_{n+1}} > 0$ for every $n = 0, 1, ..., m-1$. Let $\bar{n}$ be the maximal n such that $j_n \in J$. By construction, $n < m$. Then $h^{j_{\bar{n}}}_{j_{\bar{n}+1}} > 0$, $j_{\bar{n}} \in J$ and $j_{\bar{n}+1} \notin J$, which settles the claim.

Now consider the $(I-1)\times(I-1)$ submatrix of $(Id-H)$ in which the d-th row and the d-th column have been cancelled. Denote this submatrix by $M := (m_{ij})_{i,j \in \mathcal{I} \setminus \{d\}}$. We now claim that M has a *quasi-dominant diagonal* as defined in Murata (1977, chapter 1, Definition 6). To see this it suffices to show that for any non-empty subset $J \subset \mathcal{I} \setminus \{d\}$,

$$| m_{ij} | \quad \geq \sum_{l \in J \setminus \{j\}} | m_{lj} | \quad \text{for all} \quad j \in J,$$

with at least one inequality being strict.

Rewriting this condition yields the condition

$$1 - h_j^j \geq \sum_{l \in J \setminus \{j\}} h_l^j, \quad \text{for} \quad j \in J,$$

with one inequality being strict. Now observe that by construction for any $j \in \mathcal{I}$,

$$1 - h_j^j = \sum_{l \in \mathcal{I} \setminus \{j\}} h_l^j \geq \sum_{l \in J \setminus \{j\}} h_l^j, \quad \text{for any} \quad J \subset \mathcal{I}.$$

It therefore remains to be shown that for any $J \subset \mathcal{I} \setminus \{d\}$ there is at least one j such that

$$1 - h_j^j > \sum_{l \in J \setminus \{j\}} h_l^j.$$

But this claim follows from the definition of d. As shown above, for any $J \subset \mathcal{I} \setminus \{d\}$ there is some $j \in J$ such that h_i^j for some $i \notin J$. Hence, for this j,

$$\sum_{l \in J \setminus \{j\}} m_{lj} = \sum_{l \in J \setminus \{j\}} h_l^j < \sum_{l \in \{J \cup \{i\}\} \setminus \{j\}} h_l^j \leq \sum_{l \in \mathcal{I} \setminus \{j\}} h_l^j = 1 - h_j^j = m_{jj},$$

which proves that M has a quasi-dominant diagonal.

A result of Uekawa (1971) now implies that this matrix has full rank $I - 1$ (see Murata (1977), chapter 1, Theorem 21).

□

The notion of belief connectedness given in the above definition is the weakest notion of connectedness we can provide, which still implies homogeneous beliefs. It is easily seen that the notion of belief connectedness is implied by the connectedness of the directed communication graph (henceforth referred to as *'strong connectedness'* of the communication network), and that it implies the connectedness of the undirected communication graph (*'weak connectedness'*). It is also easily seen that weak connectedness is however not sufficient to imply the homogeneity of beliefs. To see this, consider an economy with three agents in which the first agent puts equal weights on all three agents beliefs while the other two agents only belief in themselves. Obviously , even with communication, the last two agents can have different beliefs although due to the beliefs of the first agent the undirected communication graph is connected.

In the special case where agents are connected if they meet in the assembly of shareholders (i.e. $h_j^i > 0$ if and only if $\delta_k^i \delta_k^j > 0$ for some $k = 1, \ldots, K$), the matrix H is symmetric in the sense that $h_j^i > 0$ if and only if $h_i^j > 0$. In this case, the definitions of "strong", "weak", and "belief" connectedness are equivalent.

Note that homogeneity of beliefs implies that the production decision of the firm is independent from the composition of its set of shareholders, i.e. the Fisher–Separation–Principle holds. Moreover in this case all firms evaluate their production plans according to the same price vector so that production efficiency is also obtained by the homogeneity of beliefs.

5 Pareto–efficiency of Stock Market Equilibria with financial innovation and communication

In this section we will show that using our previous arguments Pareto–efficiency of stock market economies can be obtained if and only if there are 'sufficiently many' firms.[12] The number of firms is sufficient in this sense if there are at least as many firms as there are consumers.

Note that belief connectedness itself is not sufficient to imply Pareto–efficiency if the number of firms is not sufficiently large and no relevant financial policy is possible. This claim follows from a reconsideration of our Example 1. In this example, assigning any strictly positive vector of state prices to the firm will lead to the choice of production plans which prohibit risk sharing! In particular the firms' state price vectors can be chosen to be the homogeneous beliefs which agents might hold.

Now suppose, however, that the firm in Example 1 could make two separate decisions: On the one hand it chooses a production plan such as to maximize the firms' market value , on the other hand, and completely independently, it issues a financial security such as to accomodate its shareholders spanning needs. Then complete risk sharing would be provided and a Pareto-efficient solution would be achieved.

Example 1 (continued)

Slightly modifying the assumptions, assume now that $\delta^1 = 1/4$ *and* $\delta^2 = 3/4$*, i.e. that ownership in the firm is no longer equally shared between the consumers.*[13] *We claim that the Pareto-efficient allocation*

$$\begin{aligned} \overset{*}{x}{}^1 &= \frac{1}{4}(0, 2+\varepsilon, 2+\varepsilon) \text{ and} \\ \overset{*}{x}{}^2 &= \frac{1}{4}(0, 2+3\varepsilon, 2+3\varepsilon) \end{aligned}$$

is a competitive equilibrium with relevant financial policy and communication. In fact, letting $\overset{*}{\pi}{}^i = (1,1,1)$*,* $i = 1,2$*, all consistency requirements are met. Then* $\overset{*}{y} = (-2, \varepsilon, \varepsilon)$ *and hence*

$$\begin{aligned} z^1(\overset{*}{\pi}) &= \frac{1}{4}(-2, 2) \text{ and} \\ z^2(\overset{*}{\pi}) &= \frac{1}{4}(2, -2). \end{aligned}$$

[12] It is well-known that in special cases one does not need such a lower bound on the number of firms. An important example is given by the assumption of "equilibrium spanning" introduced by Diamond (1967) and discussed in Eichberger and Harper (1997, Chapter 5.2.2). This assumption states that the production plans chosen by the firms in equilibrium span a subspace containing every firm's production set. The assumption of "equilibrium spanning" will generically be satisfied in the (very) special case where all production sets lie in some K-dimensional subspace of $\mathbb{R}^{S+1}$.

[13] Otherwise the rank condition stated in the following theorem is no longer satisfied.

Hence the firm will innovate the financial asset

$$\overset{*}{a}=\frac{1}{4}\overset{*}{z}{}^{1}+\frac{3}{4}\overset{*}{z}{}^{2}=\frac{1}{4}(1,-1)$$

But this is exactly the asset, which the consumers need to effectuate their desired net-trades, i.e.

$$(\overset{*}{x}{}^{i}_{1}-\delta^{i}\overset{*}{y}_{1}-\omega^{i}_{1})\in<\overset{*}{a}>\,.$$

$\square$

Hence, by disentangling the problems of market value maximization and of financial innovation, complete risk sharing in the economy can be obtained. The following theorem shows that this point is true in general. It is important to note here, that the following result does not impose any restrictions on S, that is on the magnitude of the number of potential states of the world. Theorem 1 is derived from the financial policy of the firm.

Theorem 1

Suppose the economy is belief connected and the matrix of ownership has rank $\triangle = I$ or rank $[\triangle^{\backslash d}-\delta^{d}\mathbb{1}]=\mathrm{I}-1$ for some $d\in\mathcal{I}$. Then competitive stock market equilibria with relevant financial policy and communication are Pareto–efficient [14].

Proof:

The result follows from the claim that under the assumptions stated a stock market equilibrium with relevant financial policy and communication is, in fact, also an Arrow-Debreu-equilibrium allocation.

To prove this claim, let

$$((\overset{*}{\pi}{}^{i},\overset{*}{x}{}^{i})_{i=1,\ldots,I},\overset{*}{Y},\overset{*}{A})$$

be a stock market equilibrium with relevant financial policy and communication (NA). From Proposition 1 we know that belief connectedness implies that

$$\overset{*}{\pi}{}^{1}=\overset{*}{\pi}{}^{2}=\ldots=\overset{*}{\pi}{}^{I}=\overset{*}{\pi}\,.$$

We claim that $\overset{*}{\pi}$ is an equilibrium state price vector for the Arrow-Debreu-model. First note, that the production decision of the firm is the same for both equilibrium concepts. Secondly, note that both concepts have the same market clearing conditions. Thus, it only remains to show that the consumption decisions of the agents remain unchanged when moving from the stock market model to the Arrow-Debreu situation. To show this it suffices to prove that the agents´ complete markets demand is spanned, i.e. that

$$z^{i}_{1}(\overset{*}{\pi})\in<\overset{*}{Y}_{1},\overset{*}{A}>\,.$$

[14] We use the following notation: $\triangle=[\delta^{1},\ldots,\delta^{I}]\in\mathbb{R}^{K\times I}$, $\triangle^{\backslash d}=[\delta^{1},\ldots,\delta^{d-1},\delta^{d+1},\ldots,\delta^{I}]\in\mathbb{R}^{K\times(I-1)}$, $A=[a^{1},\ldots,a^{K}]\in\mathbb{R}^{S\times K}$, $Z_{1}=[z^{1}_{1},\ldots,z^{I}_{1}]\in\mathbb{R}^{S\times I}$, $Z^{\backslash d}_{1}=[z^{1}_{1},\ldots,z^{d-1}_{1},z^{d+1}_{1},\ldots,z^{I}]\in\mathbb{R}^{S\times(I-1)}$, $\mathbb{1}=(1,\ldots,1)\in\mathbb{R}^{\mathrm{I}-1}$.

In the first case this claim follows from the equilibrium condition

$$\overset{*}{A}=\overset{*}{Z}_1 \cdot \Delta^T$$

by observing that under the assumptions made the matrix of share ownership Δ^T has a right inverse.

To see the second case, suppose that there exists some $d \in \mathcal{I}$ such that rank $[\triangle^{\backslash d} - \delta^d \mathrm{1\!I}] = \mathrm{I} - 1$. Since then $\overset{*}{z}^i$ corresponds to the equilibrium net trades in the complete markets model, it follows that $z_1^d(\overset{*}{\pi}) = -\sum\limits_{i \neq d} z_1^i(\overset{*}{\pi})$. Hence firm k's financial policy can be written as

$$\begin{aligned} a^k &= \sum_{i=1}^{I} \delta_k^i z_1^i(\overset{*}{\pi}) \\ &= \sum_{i \neq d} (\delta_k^i - \delta_k^d) z_1^d(\overset{*}{\pi}), k = 1, \ldots, K. \end{aligned}$$

Written more compactly, this is:

$$\begin{aligned} A = &\quad Z_1^{\backslash d}[\triangle^{\backslash d} - \delta^d \mathrm{1\!I}]. \quad \text{Solving for } \mathrm{Z}_1^{\backslash \mathrm{d}} \text{ we arrive at:} \\ Z_1^{\backslash d} = &\quad A[\triangle^{\backslash d} - \delta^d \mathrm{1\!I}]^{\mathrm{T}} \{[\triangle^{\backslash \mathrm{d}} - \delta^{\mathrm{d}} \mathrm{1\!I}][\triangle^{\backslash \mathrm{d}} - \delta^{\mathrm{d}} \mathrm{1\!I}]^{\mathrm{T}}\}^{-1} \end{aligned}$$

that is to say $z_1^i \in < A >, \quad i \neq d$.

Since $z_1^d(\overset{*}{\pi}) = -\sum\limits_{i \neq d} z_1^i(\overset{*}{\pi})$, from this it also follows that $z_1^d \in < A >$, and we obtain that $z_1^i(\overset{*}{\pi}) \in < A >, \quad i = 1, \ldots, I$.

□

Note that, generically in δ_k^i, rg$[\triangle^{\backslash d} - \delta^d \mathrm{1\!I}] = \mathrm{I} - 1$ provided $K \geq I - 1$. Hence a sufficient assumption to obtain both belief connectedness and full rank is $K \geq I - 1$ provided consumers' initial shares are chosen from some generic subset of the set of all positive $K \times K$ matrices..

For an intuition of this claim reconsider Example 1 once again. In the case where $\delta^1 = \delta^2 = \frac{1}{2}$, the economy is belief connected, rg$(\triangle) = I - 1$ but still equilibria are not Pareto–efficient because then $\overset{*}{a}= \delta^1 z_1^1(\overset{*}{\pi}) + \delta^2 z_1^2(\overset{*}{\pi}) = \frac{1}{2}(z_1^1(\overset{*}{\pi}) + z_1^2(\overset{*}{\pi})) = 0$. This choice of initial shares violates the rank condition rank $[\triangle^{\backslash d} - \delta^d \mathrm{1\!I}] = \mathrm{I} - 1$ because in this case $[\triangle^{\backslash d} - \delta^d \mathrm{1\!I}] = (1 - 2\delta^{\mathrm{d}})$ which is 0 for both agents $d = 1, 2$. However, this choice is clearly exceptional and as mentioned above any other choice would lead to full Pareto–efficiency because $\overset{*}{a}= (1 - 2\delta^1) z_1^1(\overset{*}{\pi})$.

For an interpretation of Theorem 1 note that in our Example 2 with I firms using an active financial policy the communication of beliefs *selects* the Pareto–efficient equilibrium. Hence in this setting efficient financial innovation can be seen as a coordination problem (Citanna and Villanacci (1996)) which we solve by introducing communication. Before closing note that the condition $K \geq I - 1$ is not tight since to some extent agents‘ excess demands can also be spanned by the stock markets.

6 Conclusion

When markets are incomplete the firm's production decision has two effects on consumers: it changes the market value of their shares and it changes their risk sharing opportunities. To disentangle these two conflicting objectives we allow the firm to choose some active financial policy, i.e. to issue assets for which the production plan is used as collateral.

We assume that consumers hold beliefs about the profitability of alternative production plans. Depending on the degree of heterogeneity of beliefs the resulting equilibria can be Pareto–efficient or Pareto–inefficient. The question of Pareto–efficiency then becomes a coordination problem which we solve by communication of beliefs. Our main results show that stock market equilibria with active financial policy and communication are Pareto–efficient provided there are at least as many firms as there are consumers.

Further research should investigate our idea in multi-commodity and multi-period models in which interesting observations on the theory of the firm have recently been made by Bonnisseau and Lachiri (2004 and 2006) and by Dierker (2012).

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